

Chapter 12. Limits and Derivatives

Question-1

Find the indicated limit: $\lim_{x \rightarrow 1} \frac{x^2 + 2x + 5}{x^2 + 1}$

Solution:

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^2 + 2x + 5}{x^2 + 1} &= \frac{1 + 2 + 5}{1 + 1} \text{ by direct substitution} \\ &= \frac{8}{2} =\end{aligned}$$

Question-2

Find the indicated limit: $\lim_{x \rightarrow 2} \frac{x - 2}{\sqrt{2} - x}$

Solution:

$$\lim_{x \rightarrow 2} \frac{x - 2}{\sqrt{2} - x} = \lim_{x \rightarrow 2} \frac{-(2 - x)}{\sqrt{2} - x} = \lim_{x \rightarrow 2} -(\sqrt{2} - x) = 0$$

Question-3

Find the indicated limit: $\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$

Solution:

$$\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{h(2x+h)}{h} = \lim_{h \rightarrow 0} (2x+h) = 2x$$

Question-4

Find the indicated limit: $\lim_{x \rightarrow 1} \frac{x^m - 1}{x - 1}$

Solution:

$$\lim_{x \rightarrow 1} \frac{x^m - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{x^m - 1^m}{x - 1} = m(1)^{m-1} = m$$

Question-5

Find the indicated limit: $\lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x-1} - \sqrt{2}}$

Solution:

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x-1} - \sqrt{2}} &= \lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x-1} - \sqrt{2}} \times \frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{x-2} + \sqrt{2}} \times \frac{\sqrt{2x+1} + 3}{\sqrt{2x+1} + 3} \\&= \lim_{x \rightarrow 4} \frac{(2x+1-9)}{(x-2-2)} \frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{2x+1} + 3} \\&= \lim_{x \rightarrow 4} \frac{2x-8}{x-4} \frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{2x+1} + 3} \\&= \lim_{x \rightarrow 4} 2 \left\{ \frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{2x+1} + 3} \right\} \\&= 2 \frac{2\sqrt{2}}{6} \\&= \frac{2\sqrt{2}}{3}\end{aligned}$$

Question-6

Find the indicated limit: $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + p^2} - p}{\sqrt{x^2 + q^2} - q}$

Solution:

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + p^2} - p}{\sqrt{x^2 + q^2} - q}$$

$$= \lim_{x \rightarrow 0} \frac{\sqrt{x^2 + p^2} - p}{\sqrt{x^2 + q^2} - q} \times \frac{\sqrt{x^2 + q^2} + q}{\sqrt{x^2 + q^2} + q} \times \frac{\sqrt{x^2 + p^2} + p}{\sqrt{x^2 + p^2} + p}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 + p^2 - p^2}{x^2 + q^2 - q^2} \times \frac{\sqrt{x^2 + p^2} + p}{\sqrt{x^2 + q^2} + q}$$

$$= \lim_{x \rightarrow 0} \frac{2q}{2p} = \frac{q}{p}$$

Question-7

Find the indicated limit: $\lim_{x \rightarrow a} \frac{\sqrt[m]{x} - \sqrt[m]{a}}{x - a}$

Solution:

$$\lim_{x \rightarrow a} \frac{\sqrt[m]{x} - \sqrt[m]{a}}{x - a} = \lim_{x \rightarrow a} \frac{x^{1/m} - a^{1/m}}{x - a} = \frac{1}{m} a^{1/m - 1} \text{ (na}^n - 1 \text{ formula)}$$

Question-8

Find the indicated limit: $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$

Solution:

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1} &= \lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{x^{1/2} - 1} \\&= \lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{x - 1} \times \frac{x - 1}{x^{1/2} - 1} = \lim_{x \rightarrow 1} \frac{x^{1/3} - x^{1/3}}{x - 1} \times \frac{x - 1}{x^{1/2} - 1^{1/2}} \\&= \frac{1}{3}(1)^{1/3-1} \times \frac{1}{\frac{1}{2}(1)^{1/2-1}} = \frac{1/3}{1/2} = \frac{2}{3}\end{aligned}$$

Question-9

Find the indicated limit: $\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{x}$

Solution:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{x} \times \frac{\sqrt{1+x+x^2} + 1}{\sqrt{1+x+x^2} + 1} = \lim_{x \rightarrow 0} \frac{1+x+x^2 - 1}{x[\sqrt{1+x+x^2} + 1]} \\&= \lim_{x \rightarrow 0} \frac{x(x+1)}{x(\sqrt{1+x+x^2} + 1)} \\&= \frac{1}{\sqrt{1+1}} = \frac{1}{2}\end{aligned}$$

Question-10

Find the indicated limit: $\lim_{x \rightarrow 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2}$

Solution:

$$\lim_{x \rightarrow 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2} = \lim_{x \rightarrow 0} \left(\frac{\sin x/3}{x/3} \right)^2 \frac{(x/3)^2}{x^2} = 1 \times \frac{1}{9} = \frac{1}{9}$$

Question-11

Find the indicated limit: $\lim_{x \rightarrow 0} \frac{\sin(a+x) - \sin(a-x)}{x}$

Solution:

$$\lim_{x \rightarrow 0} \frac{\sin(a+x) - \sin(a-x)}{x} = \lim_{x \rightarrow 0} \frac{2 \cos a \sin x}{x} = \lim_{x \rightarrow 0} 2 \cos a \left(\frac{\sin x}{x} \right) = 2 \cos a$$

Question-12

Find the indicated limit: $\lim_{x \rightarrow 0} \frac{\log(1+ax)}{x}$

Solution:

$$\lim_{x \rightarrow 0} \frac{\log(1+ax)}{x} = \lim_{x \rightarrow 0} \frac{ax - \frac{(ax)^2}{2} + \frac{(ax)^3}{3} - \dots}{x} = \lim_{x \rightarrow 0} a - \frac{a^2}{2}x + \frac{a^3}{3}x^2 - \dots = a$$

Question-13

Find the indicated limit: $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+5}$

Solution:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+5} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)^5 = e \cdot (1)^5 = e$$

Question-14

Evaluate the left and right limits of $f(x) = \frac{x^3 - 27}{x - 3}$ at $x = 3$. Does the limit of $f(x)$ as $x \rightarrow 3$ exist? Justify your answer.

Solution:

$$\lim_{x \rightarrow 3} \left(\frac{x^3 - 27}{x - 3} \right)$$

Let $x = 3 + h$

$$\text{Then } \lim_{h \rightarrow 0} \left(\frac{(3+h)^3 - 27}{(3+h) - 3} \right) = \left(\frac{9h^2 + 27h + h^3}{h} \right) = \lim_{h \rightarrow 0} 27 + 9h + h^2 = 27$$

$$\lim_{h \rightarrow 3} \frac{x^3 - 27}{x - 3}$$

Let $x = 3 - h$

$$\text{Then } \lim_{h \rightarrow 0} \frac{(3-h)^3 - 27}{(3-h) - 3} = \lim_{h \rightarrow 0} \frac{-27h^2 + 9h^2 - h^3}{-h} = 27$$

$$\text{Also, } \lim_{x \rightarrow 3} \frac{x^3 - 27}{x - 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x^2 + 3x + 9)}{x - 3} = 9 + 9 + 9 = 27$$

Question-15

Find the positive integer n such that $\lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3} = 108$.

Solution:

$$\lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3} = 108$$

$$n 3^{n-1} = 108$$

$$\text{Put } n = 4, \text{ then } 4 \cdot 3^3 = 4 \times 27 = 108$$

$$\therefore n = 4$$

Question-16

Evaluate $\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ [Hint: Take e^x or $e^{\sin x}$ as common factor in numerator]

Solution:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x} &= \lim_{x \rightarrow 0} \frac{e^x \left(1 - \frac{e^{\sin x}}{e^x}\right)}{(x - \sin x)} \\&= \frac{e^x(1 - e^{\sin x - x})}{(x - \sin x)} = \frac{e^x(1 - e^{-(x - \sin x)})}{(x - \sin x)} \\&= e^x \left[\frac{1 - (1 - (x - \sin x) + (x - \sin x)^2 + \dots)}{(x - \sin x)} \right] \\&= e^x \left[\frac{(x - \sin x) - (x - \sin x)^2 + \dots}{(x - \sin x)} \right] \\&= \lim_{x \rightarrow 0} e^x [1 - (x - \sin x) \dots] = e^0 = 1\end{aligned}$$

Question-17

If $f(x) = \frac{ax^2 + b}{x^2 - 1}$, $\lim_{x \rightarrow 0} f(x) = 1$ and $\lim_{x \rightarrow \infty} f(x) = 1$

Solution:

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{ax^2 + b}{x^2 - 1} = b/-1 = -b = 1$$

$$\therefore b = -1$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{a + \frac{b}{x^2}}{1 - \frac{1}{x^2}} = a = 1$$

$$\therefore a = 1$$

$$\therefore f(x) = \frac{x^2 - 1}{x^2 - 1} = 1$$

$$f(2) = 1 \text{ and } f(-2) = 1$$

Question-18

Evaluate $\lim_{x \rightarrow 0} \frac{|x|}{x}$ and $\lim_{x \rightarrow 0} \frac{|x|}{x}$. What can you say about $\lim_{x \rightarrow 0} \frac{|x|}{x}$?

Solution:

Let $f(x) = \frac{x}{|x|}$ where $|x| = x \quad x \geq 0$
 $\quad \quad \quad = -x \quad x < 0$

$$\text{Then, } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{x}{|x|} = \lim_{x \rightarrow 0^+} \frac{x}{x} = 1$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{x}{|x|} = \lim_{x \rightarrow 0^-} \frac{x}{-x} = -1$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$$

$\therefore \lim_{x \rightarrow 0} \frac{x}{|x|}$ does not exist.

Question-19

Compute $\lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$, $a, b > 0$. Hence evaluate $\lim_{x \rightarrow 0} \frac{5^x - 6^x}{x}$.

Solution:

$$\lim_{x \rightarrow 0} \frac{a^x - b^x}{x} = \lim_{x \rightarrow 0} \frac{(a^x - 1) - (b^x - 1)}{x} = \lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x} \right) - \lim_{x \rightarrow 0} \left(\frac{b^x - 1}{x} \right) = \log a - \log b = \log \left(\frac{a}{b} \right)$$

$$\therefore \lim_{x \rightarrow 0} \frac{5^x - 6^x}{x} = \log \left(\frac{5}{6} \right)$$

Question-20

Without using the series expansion of $\log(1+x)$, prove that $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

Solution:

$$\lim_{x \rightarrow 0} \frac{\log(1+x)}{x}$$

Let $y = \log(1+x)$ Then as $x \rightarrow 0$, $y \rightarrow 0$

$$\lim_{y \rightarrow 0} \frac{y}{e^y - 1} = \lim_{y \rightarrow 0} \frac{y}{1 + y + \frac{y^2}{2} + \dots - 1} = \lim_{y \rightarrow 0} \frac{y}{y \left(1 + \frac{y}{2} + \frac{y^2}{3} + \dots \right)} = \frac{1}{1} = 1$$

Question-21

Differentiate the following with respect to x:

(i) $x^7 + e^x$

(ii) $\log_7 x + 200$

(iii) $3 \sin x + 4 \cos x - e^x$

(iv) $e^x + 3 \tan x + \log x^6$

(v) $\sin 5 + \log_{10} x + 2 \sec x$

(vi) $x^{-3/2} + 8e + 7 \tan x$

(vii) $\left(x + \frac{1}{x}\right)^3$

(viii) $\frac{(x-3)(2x^2-4)}{x}$

Solution:

(i) $y = x^7 + e^x$

$$\frac{dy}{dx} = 7x^6 + e^x$$

(ii) $y = \log_7 x + 200$

$$= \log_e x \cdot \log_{10} e + 200$$

$$\frac{dy}{dx} = \log_{10} e \left(\frac{1}{x}\right)$$

$$(iii) y = 3 \sin x + 4 \cos x - e^x$$

$$\frac{dy}{dx} = 3 \cos x - 4 \sin x - e^x$$

$$(iv) y = e^x + 3 \tan x + 6 \log x$$

$$\frac{dy}{dx} = e^x + 3 \sec^2 x + \frac{6}{x}$$

$$(v) y = \sin 5 + \log_{10} x + 2 \sec x$$

$$= \sin 5 + \log_e x \log_{10} e + 2 \sec x$$

$$\frac{dy}{dx} = 0 + \frac{\log_{10} e}{x} + 2 \sec x \tan x$$

$$(vi) y = x^{-3/2} + 8e + 7 \tan x$$

$$\frac{dy}{dx} = -\frac{3}{2} x^{-5/2} + 7 \sec^2 x$$

$$(vii) y = \left(x + \frac{1}{x}\right)^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} = x^3 + 3x + 3x^{-1} + x^{-3}$$

$$\frac{dy}{dx} = 3x^2 + 3 - 3x^{-2} - 3x^{-4}$$

$$(viii) \frac{(x-3)(2x^2-4)}{x} = \frac{2x^3-6x^2-4x+12}{x}$$

$$y = 2x^2 - 6x - 4 + \frac{12}{x}$$

$$\frac{dy}{dx} = 4x - 6 - \frac{12}{x^2}$$

Question-22

Differentiate the following function using quotient rule.

$$\frac{1}{ax^2 + bx + c}$$

Solution:

$$\text{Let } y = \frac{1}{ax^2 + bx + c}$$

$$\frac{dy}{dx} = \frac{(ax^2 + bx + c) \cdot 0 - 1(2ax + b)}{(ax^2 + bx + c)^2} = - \frac{(2ax + b)}{(ax^2 + bx + c)^2}$$

Question-23

Differentiate the following function using quotient rule.

$$\frac{\tan x + 1}{\tan x - 1}$$

Solution:

$$\text{Let } y = \frac{\tan x + 1}{\tan x - 1}$$

$$\frac{dy}{dx} = \frac{(\tan x - 1)(\sec^2 x) - (\tan x + 1)\sec^2 x}{(\tan x - 1)^2} = \frac{-2\sec^2 x}{(\tan x - 1)^2}$$

Question-24

Differentiate the following function using quotient rule.

$$\frac{\sin x + x \cos x}{x \sin x - \cos x}$$

Solution:

$$\text{Let } y = \frac{\sin x + x \cos x}{x \sin x - \cos x}$$

$$\frac{dy}{dx} = \frac{(x \sin x - \cos x)(\cos x - x \sin x + \cos x) - (\sin x + x \cos x)(x \cos x + \sin x + \sin x)}{(x \sin x - \cos x)^2}$$

$$\begin{aligned} &= \frac{x \sin x \cos x - \cos^2 x - x^2 \sin^2 x + x \sin x \cos x + x \sin \cos x - \cos^2 x}{(x \sin x - \cos x)^2} \\ &= \frac{-x \sin x \cos x - x^2 \cos^2 x - \sin^2 x + x \sin x \cos x - \sin^2 x - x \sin x \cos x}{(x \sin x - \cos x)^2} \end{aligned}$$

$$= \frac{-2 \cos^2 x - 2 \sin^2 x - x^2 \sin^2 x + x^2 \cos^2 x}{(x \sin x - \cos x)^2}$$

$$= \frac{-(2 + x^2)}{(x \sin x - \cos x)^2}$$

Question-25

Differentiate the following function using quotient rule.

Solution:

$$\text{Let } y = \frac{\log x^2}{e^x} = \frac{2 \log x}{e^x}$$

$$\frac{dy}{dx} = \frac{e^x \frac{2}{x} - 2 \log x \cdot e^x}{(e^x)^2} = \frac{2e^x \left(\frac{1}{x} - \log x \right)}{(e^x)^2} = \frac{2 \left(\frac{1}{x} - \log x \right)}{e^x} = e^{-x} \left(\frac{2}{x} - 2 \log x \right)$$

Question-26

Differentiate the following function with respect to x.

$\log(\sin x)$

Solution:

$$y = \log(\sin x)$$

$$\text{Let } u = \sin x$$

$$\frac{du}{dx} = \cos x$$

$$y = \log u$$

$$\frac{dy}{du} = \frac{1}{u} = \frac{1}{\sin x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \cdot \cos x = \frac{1}{\sin x} \cos x = \cot x$$

Question-27

Differentiate the following function with respect to x.

$$e^{\sin x}$$

Solution:

$$y = e^{\sin x}$$

Put $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$y = e^u$$

$$\frac{dy}{du} = e^u = e^{\sin x}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^{\sin x} \cdot \cos x$$

Question-28

Differentiate the following function with respect to x.

$$\sqrt{1 + \cot x}$$

Solution:

$$y = \sqrt{1 + \cot x}$$

Put $u = 1 + \cot x$

$$\frac{du}{dx} = -\operatorname{cosec}^2 x$$

$$y = u^{1/2}$$

$$\frac{dy}{du} = \frac{1}{2} u^{-1/2} = \frac{1}{2} (1 + \cot x)^{-1/2}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2} (1 + \cot x)^{-1/2} (-\operatorname{cosec}^2 x)$$

Question-29

Differentiate the following function with respect to x.
 $\tan(\log x)$

Solution:

$$y = \tan(\log x)$$

Put $u = \log x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$y = \tan u$$

$$\frac{dy}{du} = \sec^2 u$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2 u \cdot \frac{1}{x} = \frac{\sec^2(\log x)}{x}$$

Question-30

Differentiate the following function with respect to x.

$$\frac{e^{bx}}{\cos(ax+b)}$$

Solution:

$$y = \frac{e^{bx}}{\cos(ax+b)}$$

$$\frac{dy}{dx} = \frac{\cos(ax+b) \cdot \frac{d}{dx}(e^{bx}) - e^{bx} \frac{d}{dx}(\cos(ax+b))}{\cos^2(ax+b)}$$

$$= \frac{\cos(ax+b) \cdot e^{bx} \cdot b + e^{bx} \cdot \sin(ax+b) \cdot a}{\cos^2(ax+b)}$$

$$= \frac{e^{bx}(b \cos(ax+b) + a \sin(ax+b))}{\cos^2(ax+b)}$$

Question-31

Differentiate the following function with respect to x.

$$\log \sec \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

Solution:

$$y = \log \sec \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

$$\text{Put } u = \frac{\pi}{4} + \frac{x}{2}$$

$$\frac{du}{dx} = \frac{1}{2}$$

$$y = \log \sec u$$

$$y = \log v$$

$$\text{Put } v = \sec u$$

$$\frac{dv}{du} = \sec u \tan u$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{v} \cdot \sec \left(\frac{\pi}{4} + \frac{x}{2} \right) \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \cdot \frac{1}{2}$$

$$= \frac{1}{\sec \left(\frac{\pi}{4} + \frac{x}{2} \right)} \sec \left(\frac{\pi}{4} + \frac{x}{2} \right) \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \cdot \frac{1}{2}$$

$$= \frac{1}{2} \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

Question-32

Differentiate the following function with respect to x.

$$\log \sin(e^x + 4x + 5)$$

Solution:

$$y = \log \sin(e^x + 4x + 5)$$

$$\frac{dy}{dx} = \frac{1}{\sin(e^x + 4x + 5)} \cos(e^x + 4x + 5)(e^x + 4)$$

$$= (e^x + 4) \frac{\cos(e^x + 4x + 5)}{\sin(e^x + 4x + 5)}$$

$$= (e^x + 4) \cot(e^x + 4x + 5)$$

Question-33

Differentiate the following function with respect to x.

$$\sin(x^{3/2})$$

Solution:

$$y = \sin(x^{3/2})$$

$$\text{Put } u = x^{3/2}$$

$$y = \sin u$$

$$\frac{du}{dx} = \frac{3}{2} x^{1/2}$$

$$\frac{dy}{du} = \cos u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot \frac{3}{2} x^{1/2} = \cos x^{3/2} \left(\frac{3}{2} x^{1/2} \right)$$

Question-34

Differentiate the following function with respect to x.

$$\cos(\sqrt{x})$$

Solution:

$$y = \cos u$$

$$\text{Put } u = \sqrt{x}$$

$$\frac{dy}{du} = -\sin u$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin u \cdot \frac{1}{2\sqrt{x}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Question-35

Differentiate the following function with respect to x.

$$e^{\sin(\log x)}$$

Solution:

$$y = e^{\sin(\log x)}$$

Put $u = \log x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$y = e^{\sin u}$$

Put $v = \sin u$

$$\frac{dv}{du} = \cos u = \cos(\log x)$$

Put $y = e^v$

$$\frac{dy}{dv} = e^v$$

$$\text{Hence } \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = e^v \cdot \cos(\log x) \cdot \frac{1}{x} = e^{\sin(\log x)} \cos(\log x) \cdot \frac{1}{x}$$

Question-36

Find the indicated limit: $\lim_{x \rightarrow 1} \frac{x^2 + 2x + 5}{x^2 + 1}$

Solution:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 + 2x + 5}{x^2 + 1} &= \frac{1 + 2 + 5}{1 + 1} \text{ by direct substitution} \\ &= \frac{8}{2} = 4 \end{aligned}$$

Question-37

Find the indicated limit: $\lim_{x \rightarrow 2} \frac{x-2}{\sqrt{2-x}}$

Solution:

$$\lim_{x \rightarrow 2} \frac{x-2}{\sqrt{2-x}} = \lim_{x \rightarrow 2} \frac{-(2-x)}{\sqrt{2-x}} = \lim_{x \rightarrow 2} -(\sqrt{2-x}) = 0$$

Question-38

Find the indicated limit: $\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$

Solution:

$$\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{h(2x+h)}{h} = \lim_{h \rightarrow 0} (2x+h) = 2x$$

Question-39

Find the indicated limit: $\lim_{x \rightarrow 1} \frac{x^m - 1}{x - 1}$

Solution:

$$\lim_{x \rightarrow 1} \frac{x^m - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{x^m - 1^m}{x - 1} = m(1)^{m-1} = m$$

Question-40

Find the indicated limit: $\lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x-1} - \sqrt{2}}$

Solution:

$$\begin{aligned} \lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x-1} - \sqrt{2}} &= \lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - 3}{\sqrt{x-1} - \sqrt{2}} \times \frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{x-2} + \sqrt{2}} \times \frac{\sqrt{2x+1} + 3}{\sqrt{2x+1} + 3} = \lim_{x \rightarrow 4} \frac{(2x+1-9)}{(x-2-2)} \frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{2x+1} + 3} \\ &= \lim_{x \rightarrow 4} \frac{2x-8}{x-4} \frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{2x+1} + 3} \\ &= \lim_{x \rightarrow 4} 2 \left(\frac{\sqrt{x-2} + \sqrt{2}}{\sqrt{2x+1} + 3} \right) \\ &= 2 \frac{2\sqrt{2}}{6} \\ &= \frac{2\sqrt{2}}{3} \end{aligned}$$

Question-41

Find the indicated limit: $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + p^2} - p}{\sqrt{x^2 + q^2} - q}$

Solution:

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sqrt{x^2 + p^2} - p}{\sqrt{x^2 + q^2} - q} \\ &= \lim_{x \rightarrow 0} \frac{\sqrt{x^2 + p^2} - p}{\sqrt{x^2 + q^2} - q} \times \frac{\sqrt{x^2 + q^2} + q}{\sqrt{x^2 + q^2} + q} \times \frac{\sqrt{x^2 + p^2} + p}{\sqrt{x^2 + p^2} + p} \\ &= \lim_{x \rightarrow 0} \frac{x^2 + p^2 - p^2}{x^2 + q^2 - q^2} \times \frac{\sqrt{x^2 + p^2} + p}{\sqrt{x^2 + q^2} + q} \\ &= \lim_{x \rightarrow 0} \frac{2q}{2p} = \frac{q}{p} \end{aligned}$$

Question-42

Find the indicated limit: $\lim_{x \rightarrow a} \frac{n\sqrt[n]{x} - n\sqrt[n]{a}}{x - a}$

Solution:

$$\lim_{x \rightarrow a} \frac{n\sqrt[n]{x} - n\sqrt[n]{a}}{x - a} = \lim_{x \rightarrow a} \frac{x^{1/n} - a^{1/n}}{x - a} = \frac{1}{n} a^{1/n - 1} \quad (na^{n-1} \text{ formula})$$

Question-43

Find the indicated limit: $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$

Solution:

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1} &= \lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{x^{1/2} - 1} \\&= \lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{x - 1} \times \frac{x - 1}{x^{1/2} - 1} = \lim_{x \rightarrow 1} \frac{x^{1/3} - x^{1/3}}{x - 1} \times \frac{x - 1}{x^{1/2} - 1^{1/2}} \\&= \frac{1}{3} (1)^{1/3 - 1} \times \frac{1}{\frac{1}{2} (1)^{1/2 - 1}} = \frac{1/3}{1/2} = \frac{2}{3}\end{aligned}$$

Question-44

Find the indicated limit: $\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{x}$

Solution:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{x} \times \frac{\sqrt{1+x+x^2} + 1}{\sqrt{1+x+x^2} + 1} = \lim_{x \rightarrow 0} \frac{1+x+x^2 - 1}{x(\sqrt{1+x+x^2} + 1)} \\&= \lim_{x \rightarrow 0} \frac{x(x+1)}{x(\sqrt{1+x+x^2} + 1)} \\&= \frac{1}{\sqrt{1+1}} = \frac{1}{2}\end{aligned}$$

Question-45

Find the indicated limit: $\lim_{x \rightarrow 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2}$

Solution:

$$\lim_{x \rightarrow 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2} = \lim_{x \rightarrow 0} \left(\frac{\sin x / 3}{x / 3}\right)^2 \frac{(x / 3)^2}{x^2} = 1 \times \frac{1}{9} = \frac{1}{9}$$

Question-46

Find the indicated limit: $\lim_{x \rightarrow 0} \frac{\sin(a+x) - \sin(a-x)}{x}$

Solution:

$$\lim_{x \rightarrow 0} \frac{\sin(a+x) - \sin(a-x)}{x} = \lim_{x \rightarrow 0} \frac{2 \cos a \sin x}{x} = \lim_{x \rightarrow 0} 2 \cos a \left(\frac{\sin x}{x}\right) = 2 \cos a$$

Question-47

Find the indicated limit: $\lim_{x \rightarrow 0} \frac{\log(1+ax)}{x}$

Solution:

$$\lim_{x \rightarrow 0} \frac{\log(1+ax)}{x} = \lim_{x \rightarrow 0} \frac{ax - \frac{(ax)^2}{2} + \frac{(ax)^3}{3} - \dots}{x} = \lim_{x \rightarrow 0} a - \frac{a^2}{2}x + \frac{a^3}{3}x^2 - \dots$$

Question-48

Find the indicated limit: $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+5}$

Solution:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+5} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)^5 = e \cdot (1)^5 = e$$

Question-49

Evaluate the left and right limits of $f(x) = \frac{x^3 - 27}{x - 3}$ at $x = 3$. Does the limit of $f(x)$ as $x \rightarrow 3$ exist? Justify your answer.

Solution:

$$\lim_{x \rightarrow 3} \left(\frac{x^3 - 27}{x - 3} \right)$$

Let $x = 3 + h$

$$\text{Then } \lim_{h \rightarrow 0} \left(\frac{(3+h)^3 - 27}{(3+h) - 3} \right) = \left(\frac{9h^2 + 27h + h^3}{h} \right) = \lim_{h \rightarrow 0} 27 + 9h + h^2 = 27$$

$$\lim_{h \rightarrow 3} \frac{x^3 - 27}{x - 3}$$

Let $x = 3 - h$

$$\text{Then } \lim_{h \rightarrow 0} \frac{(3-h)^3 - 27}{(3-h) - 3} = \lim_{h \rightarrow 0} \frac{-27h^2 + 9h^2 - h^3}{-h} = 27$$

$$\text{Also, } \lim_{x \rightarrow 3} \frac{x^3 - 27}{x - 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x^2 + 3x + 9)}{x - 3} = 9 + 9 + 9 = 27$$

Question-50

Find the positive integer n such that $\lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3} = 108$.

Solution:

$$\lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3} = 108$$

$$n 3^{n-1} = 108$$

$$\text{Put } n = 4, \text{ then } 4 \cdot 3^3 = 4 \times 27 = 108$$

$$\therefore n = 4$$

Question-51

Evaluate $\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ [Hint: Take e^x or $e^{\sin x}$ as common factor in numerator]

Solution:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x} &= \lim_{x \rightarrow 0} \frac{e^x \left(1 - \frac{e^{\sin x}}{e^x} \right)}{(x - \sin x)} \\&= \frac{e^x (1 - e^{\sin x - x})}{(x - \sin x)} = \frac{e^x (1 - e^{-(x - \sin x)})}{(x - \sin x)} \\&= e^x \left[\frac{1 - (1 - (x - \sin x) + (x - \sin x)^2 + \dots)}{(x - \sin x)} \right] \\&= e^x \left[\frac{(x - \sin x) - (x - \sin x)^2 + \dots}{(x - \sin x)} \right] \\&= \lim_{x \rightarrow 0} e^x [1 - (x - \sin x) \dots] = e^0 = 1\end{aligned}$$

Question-52

If $f(x) = \frac{ax^2 + b}{x^2 - 1}$, $\lim_{x \rightarrow 0} f(x) = 1$ and $\lim_{x \rightarrow \infty} f(x) = 1$

Solution:

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{ax^2 + b}{x^2 - 1} = b/-1 = -b = 1$$

$$\therefore b = -1$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{a + \frac{b}{x^2}}{1 - \frac{1}{x^2}} = a = 1$$

$$\therefore a = 1$$

$$\therefore f(x) = \frac{x^2 - 1}{x^2 - 1} = 1$$

$$f(2) = 1 \text{ and } f(-2) = 1$$

Question-53

Evaluate $\lim_{x \rightarrow 0} \frac{|x|}{x}$ and $\lim_{x \rightarrow 0} \frac{|x|}{x}$. What can you say about $\lim_{x \rightarrow 0} \frac{|x|}{x}$?

Solution:

Let $f(x) = \frac{x}{|x|}$ where $|x| = x \quad x \geq 0$
 $\quad \quad \quad = -x \quad x < 0$

Then,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{x}{|x|} = \lim_{x \rightarrow 0} \frac{x}{x} = 1$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{x}{|x|} = \lim_{x \rightarrow 0} \frac{x}{-x} = -1$$

$$\therefore \lim_{x \rightarrow 0} f(x) \neq \lim_{x \rightarrow 0} f(x)$$

$$\therefore \lim_{x \rightarrow 0} \frac{x}{|x|} \text{ does not exist.}$$

Question-54

Compute $\lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$, $a, b > 0$. Hence evaluate $\lim_{x \rightarrow 0} \frac{5^x - 6^x}{x}$.

Solution:

$$\lim_{x \rightarrow 0} \frac{a^x - b^x}{x} = \lim_{x \rightarrow 0} \frac{(a^x - 1) - (b^x - 1)}{x} = \lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x} \right) - \lim_{x \rightarrow 0} \left(\frac{b^x - 1}{x} \right) = \log a - \log b = \log \left(\frac{a}{b} \right)$$

$$\therefore \lim_{x \rightarrow 0} \frac{5^x - 6^x}{x} = \log \left(\frac{5}{6} \right)$$

Question-55

Without using the series expansion of $\log(1+x)$, prove that $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

Solution:

$$\lim_{x \rightarrow 0} \frac{\log(1+x)}{x}$$

Let $y = \log(1+x)$ Then as $x \rightarrow 0$, $y \rightarrow 0$

$$\lim_{y \rightarrow 0} \frac{y}{e^y - 1} = \lim_{y \rightarrow 0} \frac{y}{1 + y + \frac{y^2}{2} + \dots - 1} = \lim_{y \rightarrow 0} \frac{y}{y \left(1 + \frac{y}{2} + \frac{y^2}{3} + \dots \right)} = \frac{1}{1} = 1$$

Question-56

Differentiate the following with respect to x :

(i) $x^7 + e^x$

(ii) $\log_7 x + 200$

(iii) $3 \sin x + 4 \cos x - e^x$

(iv) $e^x + 3 \tan x + \log x^6$

(v) $\sin 5 + \log_{10} x + 2 \sec x$

(vi) $x^{-3/2} + 8e + 7 \tan x$

(vii) $\left(x + \frac{1}{x}\right)^3$

(viii) $\frac{(x-3)(2x^2-4)}{x}$

Solution:

(i) $y = x^7 + e^x$

$$\frac{dy}{dx} = 7x^6 + e^x$$

(ii) $y = \log_7 x + 200 = \log_e x \cdot \log_{10} e + 200$

$$\frac{dy}{dx} = \log_{10} e \left(\frac{1}{x}\right)$$

(iii) $y = 3 \sin x + 4 \cos x - e^x$

$$\frac{dy}{dx} = 3 \cos x - 4 \sin x - e^x$$

$$(iv) y = e^x + 3 \tan x + 6 \log x$$

$$\frac{dy}{dx} = e^x + 3 \sec^2 x + \frac{6}{x}$$

$$(v) y = \sin 5 + \log_{10} x + 2 \sec x$$

$$= \sin 5 + \log_e x \log_{10} e + 2 \sec x$$

$$\frac{dy}{dx} = 0 + \frac{\log_{10} e}{x} + 2 \sec x \tan x$$

$$(vi) y = x^{-3/2} + 8e + 7 \tan x$$

$$\frac{dy}{dx} = -\frac{3}{2} x^{-5/2} + 7 \sec^2 x$$

$$(vii) y = \left(x + \frac{1}{x}\right)^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} = x^3 + 3x + 3x^{-1} + x^{-3}$$

$$\frac{dy}{dx} = 3x^2 + 3 - 3x^{-2} - 3x^{-4}$$

$$(viii) \frac{(x-3)(2x^2-4)}{x} = \frac{2x^3 - 6x^2 - 4x + 12}{x}$$

$$y = 2x^2 - 6x - 4 + \frac{12}{x}$$

$$\frac{dy}{dx} = 4x - 6 - \frac{12}{x^2}$$

Question-57

Differentiate the following functions with respect to x .

(i) $e^x \cos x$

(ii) $\sqrt[n]{x} \log \sqrt{x}, x > 0$

(iii) $6 \sin x \log_{10} x + e$

(iv) $(x^4 - 6x^3 + 7x^2 + 4x + 2)(x^3 - 1)$

(v) $(a - b \sin x)(1 - 2 \cos x)$

(vi) $\operatorname{cosec} x \cdot \cot x$

(vii) $\sin^2 x$

(viii) $\cos^2 x$

(ix) $(3x^2 + 1)^2$

(x) $(4x^2 - 1)(2x + 3)$

(xi) $(3 \sec x - 4 \operatorname{cosec} x)(2 \sin x + 5 \cos x)$

(xii) $x^2 e^x \sin x$

(xiii) $\sqrt{x} e^x \cos x$

Solution:

(i) $y = e^x \cos x$

$$\frac{dy}{dx} = -e^x \sin x + \cos x e^x$$

$$(ii) y = x^{1/n} \log(\sqrt{x}) = \frac{1}{2} x^{1/n} \log x$$

$$\frac{dy}{dx} = \frac{1}{2} \left[x^{1/n} \frac{1}{x} + \log x \left(\frac{1}{n} x^{1/n-1} \right) \right] = \frac{1}{2} \left[x^{1/n-1} + \frac{1}{n} \log x \cdot x^{1/n-1} \right]$$

$$(iii) y = 6 \sin x \log_{10} x + e = 6 \sin x \log_e x \cdot \log_{10} e + e$$

$$\frac{dy}{dx} = 6 \log_{10} e \left(\sin x \cdot \frac{1}{x} + \log x \cos x \right)$$

$$(iv) y = (x^4 - 6x^3 + 7x^2 + 4x + 2)(x^3 - 1)$$

$$\frac{dy}{dx} = (x^4 - 6x^3 + 7x^2 + 4x + 2)(3x^2) + (x^3 - 1)(4x^3 - 18x^2 + 14x + 4)$$

$$(v) y = (a - b \sin x)(1 - 2 \cos x)$$

$$\frac{dy}{dx} = (a - b \sin x)(2 \sin x) + (1 - 2 \cos x)(-b \cos x)$$

$$\frac{dy}{dx} = 2a \sin x - 2b \sin^2 x - b \cos x + 2b \cos^2 x$$

$$(vi) y = \operatorname{cosec} x \cdot \cot x$$

$$\frac{dy}{dx} = -\operatorname{cosec} x \operatorname{cosec}^2 x (-\operatorname{cosec} x \cot x) = -\operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x$$

Question-58

Differentiate the following function using quotient rule.

$$\frac{5}{x^2}$$

Solution:

$$\text{Let } y = \frac{5}{x^2}$$

$$\frac{dy}{dx} = \frac{(x^2) \cdot \frac{d}{dx}(5) - 5 \frac{d}{dx}(x^2)}{(x^2)^2} = \frac{x^2 \cdot 0 - 5(2x)}{x^4} = \frac{-10x}{x^4} = \frac{-10}{x^3}$$

Question-59

Differentiate the following function using quotient rule.

$$\frac{2x-3}{4x+5}$$

Solution:

$$\text{Let } y = \frac{2x-3}{4x+5}$$

$$\frac{dy}{dx} = \frac{(4x+5) \frac{d}{dx}(2x-3) - (2x-3) \frac{d}{dx}(4x+5)}{(4x+5)^2}$$

$$= \frac{(4x+5)(2) - (2x-3)(4)}{(4x+5)^2}$$

$$= \frac{8x+10-8x+12}{(4x+5)^2}$$

$$= \frac{22}{(4x+5)^2}$$

Question-60

Differentiate the following function using quotient rule.

$$\frac{x^7-4^7}{x-4}$$

Solution:

$$\text{Let } y = \frac{x^7-4^7}{x-4}$$

$$\frac{dy}{dx} = \frac{(x-4)7x^6 - (x^7-4^7) \cdot 1}{(x-4)^2} = \frac{6x^7-28x^6+4^7}{(x-4)^2}$$

Question-61

Differentiate the following function using quotient rule.

$$\frac{\cos x + \log x}{x^2 + e^x}$$

Solution:

$$\text{Let } y = \frac{\cos x + \log x}{x^2 + e^x}$$

$$\frac{dy}{dx} = \frac{(x^2 + e^x)\left(-\sin x + \frac{1}{x}\right) - (\cos x + \log x)(2x + e^x)}{(x^2 + e^x)^2}$$

$$\frac{dy}{dx} = \frac{e^x\left(\frac{1}{x} - \sin x - \cos x - \log x\right) - 2x(\cos x + \log x) + x - x^2 \sin x}{(x^2 + e^x)^2}$$

Question-62

Differentiate the following function using quotient rule.

$$\frac{\log x - 2x^2}{\log x + 2x^2}$$

Solution:

$$\text{Let } y = \frac{\log x - 2x^2}{\log x + 2x^2}$$

$$\frac{dy}{dx} = \frac{(\log x + 2x^2)\left(\frac{1}{x} - 4x\right) - (\log x - 2x^2)\left(\frac{1}{x} + 4x\right)}{(\log x + 2x^2)^2} = \frac{4x(1 - 2\log x)}{(\log x + 2x^2)^2}$$

Question-63

Differentiate the following function using quotient rule.

$$\frac{\log x}{\sin x}$$

Solution:

$$\text{Let } y = \frac{\log x}{\sin x}$$

$$\frac{dy}{dx} = \frac{\sin x \left(\frac{1}{x} \right) - \log x \cos x}{\sin^2 x}$$

Question-64

Differentiate the following function using quotient rule.

$$\frac{1}{ax^2 + bx + c}$$

Solution:

$$\text{Let } y = \frac{1}{ax^2 + bx + c}$$

$$\frac{dy}{dx} = \frac{(ax^2 + bx + c) \cdot 0 - 1(2ax + b)}{(ax^2 + bx + c)^2} = - \frac{(2ax + b)}{(ax^2 + bx + c)^2}$$

Question-65

Differentiate the following function using quotient rule.

$$\frac{\tan x + 1}{\tan x - 1}$$

Solution:

$$\text{Let } y = \frac{\tan x + 1}{\tan x - 1}$$

$$\frac{dy}{dx} = \frac{(\tan x - 1)(\sec^2 x) - (\tan x + 1)\sec^2 x}{(\tan x - 1)^2} = \frac{-2\sec^2 x}{(\tan x - 1)^2}$$

Question-66

Differentiate the following function using quotient rule.

$$\frac{\sin x + x \cos x}{x \sin x - \cos x}$$

Solution:

$$\text{Let } y = \frac{\sin x + x \cos x}{x \sin x - \cos x}$$

$$\frac{dy}{dx} = \frac{(x \sin x - \cos x)(\cos x - x \sin x + \cos x) - (\sin x + x \cos x)(x \cos x + \sin x + \sin x)}{(x \sin x - \cos x)^2}$$

$$= \frac{x \sin x \cos x - \cos^2 x - x^2 \sin^2 x + x \sin x \cos x + x \sin \cos x - \cos^2 x}{(x \sin x - \cos x)^2}$$

$$= \frac{-2 \cos^2 x - 2 \sin^2 x - x^2 \sin^2 x + x^2 \cos^2 x}{(x \sin x - \cos x)^2}$$

$$= \frac{-(2 + x^2)}{(x \sin x - \cos x)^2}$$

Question-67

Differentiate the following function using quotient rule.

Solution:

$$\text{Let } y = \frac{\log x^2}{e^x} = \frac{2 \log x}{e^x}$$

$$\frac{dy}{dx} = \frac{e^x \frac{2}{x} - 2 \log x \cdot e^x}{(e^x)^2} = \frac{2e^x \left(\frac{1}{x} - \log x \right)}{(e^x)^2} = \frac{2 \left(\frac{1}{x} - \log x \right)}{e^x} = e^{-x} \left(\frac{2}{x} - 2 \log x \right)$$

Question-68

Differentiate the following function with respect to x.

Solution:

$$y = \log(\sin x)$$

$$\text{Let } u = \sin x$$

$$\frac{du}{dx} = \cos x$$

$$y = \log u$$

$$\frac{dy}{du} = \frac{1}{u} = \frac{1}{\sin x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \cdot \cos x = \frac{1}{\sin x} \cos x = \cot x$$

Question-69

Differentiate the following function with respect to x.

$$e^{\sin x}$$

Solution:

$$y = e^{\sin x}$$

$$\text{Put } u = \sin x$$

$$\frac{du}{dx} = \cos x$$

$$y = e^u$$

$$\frac{dy}{du} = e^u = e^{\sin x}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^{\sin x} \cdot \cos x$$

Question-70

Differentiate the following function with respect to x . $\sqrt{1 + \cot x}$

Solution:

$$y = \sqrt{1 + \cot x}$$

$$\text{Put } u = 1 + \cot x$$

$$\frac{du}{dx} = -\operatorname{cosec}^2 x$$

$$y = u^{1/2}$$

$$\frac{dy}{du} = \frac{1}{2} u^{-1/2} = \frac{1}{2} (1 + \cot x)^{-1/2}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2} (1 + \cot x)^{1/2} (-\operatorname{cosec}^2 x)$$

Question-71

Differentiate the following function with respect to x .
 $\tan(\log x)$

Solution:

$$y = \tan(\log x)$$

$$\text{Put } u = \log x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$y = \tan u$$

$$\frac{dy}{du} = \sec^2 u$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2 u \cdot \frac{1}{x} = \frac{\sec^2(\log x)}{x}$$

Question-72

Differentiate the following function with respect to x.

$$\frac{e^{bx}}{\cos(ax+b)}$$

Solution:

$$y = \frac{e^{bx}}{\cos(ax+b)}$$

$$\frac{dy}{dx} = \frac{\cos(ax+b) \cdot \frac{d}{dx}(e^{bx}) - e^{bx} \frac{d}{dx}(\cos(ax+b))}{\cos^2(ax+b)}$$

$$= \frac{\cos(ax+b) \cdot e^{bx} \cdot b + e^{bx} \cdot \sin(ax+b) \cdot a}{\cos^2(ax+b)}$$

$$= \frac{e^{bx}(b \cos(ax+b) + a \sin(ax+b))}{\cos^2(ax+b)}$$

Question-73

Differentiate the following function with respect to x.

$$\log \sec \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

Solution:

$$y = \log \sec \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

$$\text{Put } u = \frac{\pi}{4} + \frac{x}{2}$$

$$\frac{du}{dx} = \frac{1}{2}$$

$$y = \log \sec u$$

$$y = \log v$$

$$\text{Put } v = \sec u$$

$$\frac{dv}{du} = \sec u \tan u$$

$$\frac{dy}{dv} = \frac{1}{v}$$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \\
 &= \frac{1}{v} \cdot \sec\left(\frac{\pi}{4} + \frac{x}{2}\right) \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot \frac{1}{2} \\
 &= \frac{1}{\sec\left(\frac{\pi}{4} + \frac{x}{2}\right)} \sec\left(\frac{\pi}{4} + \frac{x}{2}\right) \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot \frac{1}{2} \\
 &= \frac{1}{2} \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)
 \end{aligned}$$

Question-74

Differentiate the following function with respect to x.
 $\log \sin(e^x + 4x + 5)$

Solution:

$$y = \log \sin(e^x + 4x + 5)$$

$$\frac{dy}{dx} = \frac{1}{\sin(e^x + 4x + 5)} \cos(e^x + 4x + 5)(e^x + 4)$$

$$= (e^x + 4) \frac{\cos(e^x + 4x + 5)}{\sin(e^x + 4x + 5)}$$

$$= (e^x + 4) \cot(e^x + 4x + 5)$$

Question-75

Differentiate the following function with respect to x.

$$\sin(x^{3/2})$$

Solution:

$$y = \sin(x^{3/2})$$

$$\text{Put } u = x^{3/2}$$

$$y = \sin u$$

$$\frac{du}{dx} = \frac{3}{2} x^{1/2}$$

$$\frac{dy}{du} = \cos u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot \frac{3}{2} x^{1/2} = \cos x^{3/2} \left(\frac{3}{2} x^{1/2} \right)$$

Question-76

Differentiate the following function with respect to x.

$$\cos(\sqrt{x})$$

Solution:

$$y = \cos u$$

$$\text{Put } u = \sqrt{x}$$

$$\frac{dy}{du} = -\sin u$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin u \cdot \frac{1}{2\sqrt{x}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Question-77

Differentiate the following function with respect to x.

$$e^{\sin(\log x)}$$

Solution:

$$y = e^{\sin(\log x)}$$

Put $u = \log x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$y = e^{\sin u}$$

Put $v = \sin u$

$$\frac{dv}{du} = \cos u = \cos(\log x)$$

Put $y = e^v$

$$\frac{dy}{dv} = e^v$$

$$\text{Hence } \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = e^v \cdot \cos(\log x) \cdot \frac{1}{x} = e^{\sin(\log x)} \cos(\log x) \cdot \frac{1}{x}$$